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Eulerian Opinion Dynamics with Bounded Confidence and Exogenous Inputs*

Anahita Mirtabatabaei†, Peng Jia†, and Francesco Bullo†

Abstract. The formation of opinions in a large population is governed by endogenous (interactions with peers) and exogenous (influence of media) factors. In the analysis of opinion evolution in a large population, decision making rules are often approximated with non-Bayesian "rule of thumb" methods. Adopting a non-Bayesian averaging rule, this paper focuses on an Eulerian bounded-confidence model of opinion dynamics and studies the information assimilation process resulting from exogenous inputs. In this model, a population is distributed over an opinion set, and each individual updates its opinion via (i) opinions of the population inside the individual's confidence range and (ii) the information from an exogenous input in that range. First, we establish various mathematical properties of this system's dynamics with time-varying inputs. Second, for the case of no exogenous input, we prove the convergence of the population's distribution to a sum of Dirac delta distributions. We further derive a simple sufficient condition for opinion consensus under the influence of a time-varying input. Third, regarding information assimilation, we define the *attracted population* of a constant input. For a weighted Dirac delta input and for uniformly distributed initial population, we establish an upper bound on the attracted population valid under some technical assumptions. This upper bound is an increasing function of the population's confidence bound and a decreasing function of the input's measure (i.e., the integral of input's distribution over the opinion space). Fourth, for a normally distributed input with truncated support, we conjecture that the attracted population is approximately an increasing affine function of the population's confidence bound and of the input's standard deviation; we illustrate this conjecture numerically.

Key words. opinion dynamics, distributed averaging algorithm, exogenous input, assimilation of information, mass measures, Eulerian approach

AMS subject classifications. 91C20, 91C05, 28D05, 91D30, 93A14, 91B14

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1. Introduction.

Opinion dynamics in a society is a complex process which is led to its final state by *endogenous* and *exogenous* factors. The interaction of people via in-person meetings or online social networks is an endogenous factor. One of the most influential exogenous factors is the mainstream media that acts as a real-time input owing to its easy access to the public. Quoting from [10], "After introduction and expansion of Fox News, between 1996 and 2000, it is estimated that 3–28 percent of the audience was persuaded to vote Republican." Media influences opinions by employing some well-known techniques such as repeated exposure to experts' messages. The bias in mainstream media in reference to the members of the US Congress is quantified and analyzed in [17].

Developing quantifiable descriptions of societal opinion dynamics has a long history and

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†Center for Control, Dynamical Systems, and Computation, University of California, Santa Barbara, Santa Barbara, CA 93106 (mirtabatabaei@engineering.ucsb.edu, pjia@engineering.ucsb.edu, bullo@engineering.ucsb.edu).

can be classified based on a variety of characteristics. The early references [27, 7, 14] propose models for "continuous opinion dynamics," where opinions are represented by real numbers. Such a real number represents the attitude or position of an individual in relation to an issue—for instance, "political opinions and actions or roles in economic life as a producer or consumer" [28]. Therefore, the position of an individual can vary smoothly from one extreme of the range of possible choices to the other. According to [5, 13], models of opinion dynamics can be described by either a *Lagrangian* or an *Eulerian* approach. A Lagrangian description focuses on changes in each individual's opinion, while an Eulerian description focuses on the changes in population in one opinion interval as time progresses. A Lagrangian model of opinion dynamics has been defined over a continuous [3] or discrete [12, 3, 24, 20] state space with infinite or finite number of individuals, respectively. An Eulerian model of opinion dynamics has also been defined over a continuous [19, 5, 8, 11] or discrete [2] state space depending on whether the opinion set is continuous or discretized, respectively.

A popular opinion update rule is the non-Bayesian "rule of thumb" method of averaging neighbors' opinions. This method provides a good approximation to the behavior of a large population without relying on detailed social psychological findings; see [1, 6]. In our investigation, the neighboring relationships between individuals are defined based on the concept of *bounded confidence* [15]. In other words, an individual interacts with only those individuals whose opinions are close enough to his or her own. This idea reflects (1) *filter bubbles*, a phenomenon in which websites use algorithms to show users only information that agrees with their past viewpoints [25], and (2) *selective exposure*, a psychological concept broadly defined as "behaviors that bring the communication content within reach of one's sensory apparatus" [31, 21]. Recently, Hegselmann and Krause formulated a Lagrangian bounded-confidence model (HK model) where individuals synchronously update their opinions by averaging all opinions in their confidence bound [12, 22]. Accordingly, an Eulerian HK model has been defined over a continuous state space [18, 5, 23], where a *mass distribution* over an opinion set is being updated by a *flow map* under the influence of an exogenous input. The convergence of a variation of the Eulerian HK model both in discrete and continuous time has been established in [5], where the influence weights that any two individuals assign to each other are equal, and this symmetry preserves the global opinion average during the evolution. In this context, we consider a general Eulerian HK model where the symmetric weight constraint is relaxed. Specifically, the weight an individual assigns to another is a function of the mass measure (and of the exogenous input measure) in that individual's confidence range. Since the measures on different opinions' confidence bounds are not necessarily equal, the weights assigned to different opinions are generally asymmetric, and thus the global average is not preserved. An Eulerian model of opinion dynamics with a pairwise-sequential updating procedure is studied in [8]. The effect of media on an Eulerian model of opinion formation with pairwise gossip interactions is numerically analyzed in [4]. Furthermore, two manipulation strategies that aim to increase the population with positive vote in finite time are compared in [23].

The contributions of this paper are summarized as follows.

(1) We propose a novel model for exogenous inputs in the Eulerian HK opinion dynamics model. It is well known that media's influence on the public depends on public's attitude toward it [16]. This concept for a voter decision-maker who ignores the message from an opposite political predisposition is called "partisan resistance." Accordingly, we assume that

each individual associated with one opinion receives exogenous input information only within its opinion confidence range. First, we model the input as a background weighted Dirac delta distribution centered at the opinion of an expert, where the integral of this distribution over opinion space depends on factors such as how strongly the message is being reported. Second, we observe that the increasing popularity of communication technologies such as blogging and tweeting is leading to the public rebroadcasting of the message with bias. Accordingly, we model the input as a Gaussian signal whose variance represents how much bias is inserted in the rebroadcast.

(2) We prove some important properties of the Eulerian HK model with a time-varying input. Under mild technical assumptions (the initial opinion is a finite and absolutely continuous mass distribution over the opinion set), we show that the opinion update via an Eulerian flow map has the following properties: (i) the mass distribution over the opinion space remains constant and absolutely continuous; (ii) due to the homogeneity of the confidence bounds, the flow map preserves opinions order; and (iii) the flow map is bi-Lipschitz. We further establish that if the concentration of the population in the confidence range of an individual is higher than the population concentration just outside of this range, then the density of the population at that individual's opinion increases in one iteration.

(3) Based on the above analysis, for the case of no exogenous input, we prove the convergence of the population's distribution to a sum of weighted Dirac delta distributions, whose centers are within a distance greater than the population's confidence bound apart from one another. This result validates our model by verifying a basic known property of opinion evolution, that is, the emergence of population clusters. Recent results on opinion dynamics models establish that people tend to aggregate into groups of equal-minded individuals [3, 5]. We further derive a simple sufficient condition for opinion consensus under the influence of time-varying input.

(4) In our study of information assimilation, we define the *attracted population* of a constant input, which is the portion of population that will be attracted to the input's center opinion as time goes to infinity. First, for the case of weighted Dirac delta input centered at the advertised opinion with a uniformly distributed initial population, we establish that the attracted population is an increasing function of population's confidence bound and a decreasing function of the input's measure (i.e., how strongly the message is being broadcast by media). Second, for the case of a truncated normal distribution, our simulations illustrate that the attracted population is approximately an increasing affine function of the population's confidence bound and of the input's standard deviation. This result suggests that (i) a higher biased rebroadcast of media's message by various blogs and/or (ii) a larger confidence bound result in the attraction of a larger population to the advertised message.

This paper is organized as follows. Section 2 develops the mathematical model of an Eulerian HK system with input and introduces some preliminary properties of this system. Section 3 analyzes fundamental properties of Eulerian HK systems with time-varying inputs and gives a sufficient condition for consensus. Section 4 establishes the convergence of Eulerian HK systems with no input. Section 5 presents the result of information assimilation under a single constant input. Finally, section 6 contains concluding remarks and potential future directions.

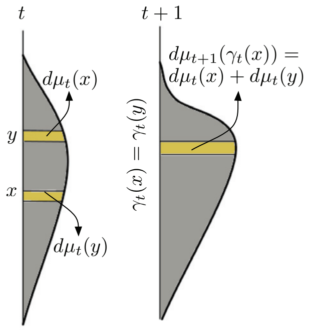

Figure 1. *A schematic illustration of push forward of a measure μ_t via the flow map γ_t.*

2. Mathematical model and preliminary results. In this section, we mathematically formulate the process of the opinion evolution in a large population by an HK model. The *mass distribution* of the population over the opinion set is represented by Radon measure μ_t (inner regular and locally finite) defined on the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ at each discrete time step t, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra. The continuous opinion space is $\mathbb{R}$, and each opinion value is denoted by an independent variable x. For any $dx \in \mathcal{B}(\mathbb{R})$, the value $\mu_t(dx)$ represents the infinitesimal population whose opinion is equal to x at time t. At $t+1$, this population updates its opinion to a new opinion $\gamma_t(x)$. The map $\gamma_t : \operatorname{supp} \mu_t \subseteq \mathbb{R} \to \mathbb{R}$ is called the *flow map* of the mass distribution, where $\operatorname{supp} \mu_t$ denotes the *support* of the measure μ_t, that is, the set of all points $x \in \mathbb{R}$ for which every open neighborhood of x has positive measure. This approach establishes an Eulerian model of opinion dynamics and is inspired by [5]. Here, the flow map is defined in compliance with the Lagrangian HK rules,

$$\gamma_t(x) = \frac{\int_{[x-r,x+r]} z\mu_t(dz) + \int_{[x-r,x+r]} z u_t(dz)}{\int_{[x-r,x+r]} \mu_t(dz) + \int_{[x-r,x+r]} u_t(dz)}, \tag{2.1}$$

where r is the population's confidence bound and u_t is the distribution of exogenous background input at time t, which is also assumed to be a Radon measure on $\mathcal{B}(\mathbb{R})$ for simplicity of analysis. Now, the mass distribution can be tracked by the recurrence relation $\mu_{t+1} = \gamma_t \# \mu_t$, where $\gamma_t \#$ denotes the *push forward* of a measure via the flow map γ_t [5]; see Figure 1. Moreover, for every $E \in \mathcal{B}(\mathbb{R})$,

$$\mu_{t+1}(E) = \mu_t(\gamma_t^{-1}(E)), \tag{2.2}$$

where $\gamma_t^{-1}(E)$ is the preimage of the set E under the flow map γ_t, though γ_t is not necessarily invertible. Therefore, $\mu_{t+1}(\mathbb{R}) = \mu_t(\mathbb{R})$; that is, the total mass distribution over the opinion space is preserved over time. Hence, via normalization, μ_t can be regarded as a probability measure.

Definition 2.1 (Eulerian HK system with input). *We call the dynamical system in which a mass distribution μ_t defined over $\mathbb{R}$ is being pushed forward by the flow map* (2.1) *under the influence of an input u_t an* Eulerian HK system with input.

In the remainder of this section, we introduce some preliminary properties of an Eulerian HK system with input. Before proceeding, let us introduce a few notations. We call a single point $x \in \mathbb{R}$ an atom with respect to a measure μ if $x \in \operatorname{supp}\mu$ and $\mu(x) > 0$. Moreover, if every μ-measurable set of positive measure contains an atom, then μ is called purely atomic or atomic for short. We denote the absolute continuity of any measure μ with respect to Lebesgue measure $\mathcal{L}^1$ by $\mu \ll \mathcal{L}^1$. The assumption of absolute continuity of the mass distributions implies that a population with measure zero is assigned to an opinion interval with zero length. In essence, the dynamics of mass distributions that contain only Dirac delta distributions can be handled by Lagrangian agent-based modeling in "microscopic" analysis [9], whereas absolutely continuous measures represent the average density of a large population over the opinion space and are employed in "macroscopic" analysis here. From here on, the smallest and largest opinions along $\operatorname{supp}\mu_t$ are denoted by $x_{\min}(t)$ and $x_{\max}(t)$, respectively, and the length of μ's support is denoted by $|\operatorname{supp}\mu|$. The flow map $x \mapsto \gamma_t(x)$ is called *bi-Lipschitz* [29] if for any $x, y \in \operatorname{supp}\mu_t$ there exists $L_t \geq 1$ such that

$$|y - x|/L_t \leq |\gamma_t(y) - \gamma_t(x)| \leq L_t|y - x|. \tag{2.3}$$

For any finite mass distribution μ_t, we define the *opinion average* over any interval $[a, b] \in \mathbb{R}$ for nonzero $\int_{[a,b]} \mu_t(dz)$ by

$$y_t([a, b]) = \frac{\int_{[a,b]} z\mu_t(dz)}{\int_{[a,b]} \mu_t(dz)}. \tag{2.4}$$

Finally, for any $\mu_t \ll \mathcal{L}^1$, there exists a Lebesgue integrable density function ρ_t such that $\mu_t(E) = \int_E \rho_t(z)dz$ for all $E \in \mathcal{B}(\mathbb{R})$.

Lemma 2.1 (bounds on opinion average). *Consider a finite mass distribution $\mu \ll \mathcal{L}^1$ whose support is a closed bounded interval in $\mathbb{R}$. Assume that its density function $\rho(x)$ satisfies $\rho(x) \in [\rho_{\min}, \rho_{\max}]$ for all $x \in \operatorname{supp}\mu$ with $0 < \rho_{\min} \leq \rho_{\max} < \infty$. Then, for all $a, b \in \operatorname{supp}\mu$, the opinion average over $[a, b]$, denoted by $y([a, b])$, is bounded as follows:*

$$a < \frac{b + a\sqrt{\rho_{\max}/\rho_{\min}}}{1 + \sqrt{\rho_{\max}/\rho_{\min}}} \leq y([a, b]) \leq \frac{a + b\sqrt{\rho_{\max}/\rho_{\min}}}{1 + \sqrt{\rho_{\max}/\rho_{\min}}} < b. \tag{2.5}$$

Proof. Here we present the upper bound of $y([a, b])$, and the similar analysis for the lower bound is skipped. First, let us introduce a *step density function* for any variable $c \in [a, b]$:

$$\rho_s(x, c) = \begin{cases} \rho_{\min} & \text{if } x \in [a, c), \\ \rho_{\max} & \text{if } x \in [c, b]. \end{cases}$$

The average opinion over $[a, b]$ for the step density function is as follows:

$$y_s([a, b], c) = \frac{(c^2 - a^2)\rho_{\min}/2 + (b^2 - c^2)\rho_{\max}/2}{(c - a)\rho_{\min} + (b - c)\rho_{\max}} =: \frac{f(c)}{g(c)}.$$

According to the first mean value theorem for integrals, one can show that, for any bounded density function $\rho : [a, b] \to [\rho_{\min}, \rho_{\max}]$, there exists $c \in [a, b]$ such that the averages of ρ and ρ_s over $[a, b]$ are equal. Therefore, $y([a, b])$ is less than or equal to the upper bound of $y_s([a, b], c)$ given any $c \in [a, b]$.

Owing to the differentiability of $y_s([a,b],c)$ with respect to c, the maximum of $y_s([a,b],c)$ over c can be computed by letting $\partial y_s([a,b],c)/\partial c$ equal zero:

$$\frac{\partial y_s([a,b],c)}{\partial c} = \frac{c(\rho_{\min} - \rho_{\max})g(c) - (\rho_{\min} - \rho_{\max})f(c)}{g^2(c)} = 0.$$

Hence, the critical point $c^* = f(c^*)/g(c^*)$ can be computed as follows:

$$2(c^{*2} - ac^*)\rho_{\min} + 2(bc^* - c^{*2})\rho_{\max} = (c^{*2} - a^2)\rho_{\min} + (b^2 - c^{*2})\rho_{\max} \Rightarrow c^* = \frac{a + b\sqrt{\rho_{\max}/\rho_{\min}}}{1 + \sqrt{\rho_{\max}/\rho_{\min}}}$$

gives the maximum of $y_s([a,b],c)$, which is equal to c^*. Together with $a < b$, we have

$$y([a,b]) \leq \max_{c\in[a,b]}\{y_s([a,b],c)\} = \frac{a + b\sqrt{\rho_{\max}/\rho_{\min}}}{1 + \sqrt{\rho_{\max}/\rho_{\min}}} < b. \qquad \blacksquare$$

Lemma 2.1 determines upper and lower bounds on μ's opinion average over an interval, which are strictly inside the interval. Accordingly, the following lemma demonstrates that if the opinion average of a time-varying mass distribution over an interval converges to the boundary of that interval, then the mass distribution converges to a Dirac delta distribution centered at that boundary. Before proceeding, consider $a, b \in \mathbb{R}$ with $a < b$ and $x, y \in \mathbb{R}_{>0}$; then it can be verified that

$$\max_{x\in[x_1,x_2], y\in[y_1,y_2]} \frac{xa + yb}{x + y} = \frac{x_1 a + y_2 b}{x_1 + y_2}. \tag{2.6}$$

Lemma 2.2 (opinion average limit). *Assume that in an Eulerian HK system with input, the mass distribution $\mu_t \ll \mathcal{L}^1$ is finite and $\operatorname{supp}\mu_t$ is a closed bounded interval and contains $[a,b] \in \mathbb{R}$ for all $t \geq 0$. The opinion average over $[a,b]$, denoted by $y_t([a,b])$, satisfies $\lim_{t\to\infty} y_t([a,b]) = a$ (or $\lim_{t\to\infty} y_t([a,b]) = b$) if and only if the mass distribution over $[a,b]$ weakly converges to a weighted Dirac delta distribution centered at a (or b, respectively).*

Proof. We prove that $\lim_{t\to\infty} y_t([a,b]) = a$ is a sufficient condition for the convergence of $\mu_t([a,b])$ to a weighted Dirac delta distribution centered at a, and the obvious proof of its necessity and the proof of the convergence to the other bound b are omitted. By contradiction, assume that the mass distribution does not converge to a Dirac delta distribution centered at a, or, equivalently, there exists point $c \in (a,b)$ such that $\mu_t([c,b]) > m$ for some $m \in \mathbb{R}_{>0}$ and an infinite number of time steps. Therefore, if we denote the density function of μ_t by ρ_t, then

$$y_t([a,b]) = \frac{\int_{[a,b]} z\mu_t(dz)}{\int_{[a,b]} \mu_t(dz)} = \frac{\int_a^b z\rho_t(z)dz}{\int_a^b \rho_t(z)dz} = \frac{\int_a^c z\rho_t(z)dz + \int_c^b z\rho_t(z)dz}{\int_a^c \rho_t(z)dz + \int_c^b \rho_t(z)dz}.$$

Since $\mu_t(\mathbb{R})$ is finite, there exists $M \in \mathbb{R}_{>0}$ such that $\mu_t([a,c]) \leq M$ for all $t \geq 0$. According to (2.6) and Lemma 2.1,

$$y_t([a,b]) > \frac{aM + cm}{M + m} = a + \frac{(c-a)m}{M + m}$$

for an infinite number of time steps, which contradicts the convergence of $y_t([a,b])$ to a. $\blacksquare$

Here, we introduce two assumptions on the initial states and inputs that are employed in sections 3 and 4.

Assumption 2.1. *For an Eulerian HK system with input, $\mu_0 \ll \mathcal{L}^1$ is finite,* $\operatorname{supp}\mu_0$ *is a closed bounded interval, and $u_t \ll \mathcal{L}^1$ for all $t \geq 0$.*

Assumption 2.2. *The set* $\operatorname{supp} u_t$ *is contained in the set* $\operatorname{supp}\mu_t$ *for all $t \geq 0$.*

The interpretation of Assumption 2.2 is that the manipulator can advertise only for opinions with nonzero population assigned to them. In other words, the manipulator disregards the opinions that nobody believes in.

3. Dynamic properties of the model.

This section analyzes some fundamental properties of Eulerian HK systems with time-varying inputs and gives a simple sufficient condition for the emergence of opinion consensus.

Theorem 3.1 (properties of an Eulerian HK system with input). *If an Eulerian HK system with input satisfies Assumption 2.1, then for any $t \geq 0$ such that $|\operatorname{supp}\mu_\tau| > 2r$ for all $\tau \leq t$,*

(i) *$\mu_t \ll \mathcal{L}^1$ is finite and* $\operatorname{supp}\mu_t$ *is a closed interval;*

(ii) *for any $x, y \in \operatorname{supp}\mu_t$, if $x < y$, then $\gamma_t(x) < \gamma_t(y)$; and*

(iii) *$x \mapsto \gamma_t(x)$ is bi-Lipschitz.*

Proof. Here we first prove that if statement (i) holds at any time $t \geq 0$, then statements (ii) and (iii) also hold at t. Next, if the three statements hold at any t, then statement (i) holds at $t+1$. Finally, since μ_0 satisfies statement (i), the three statements hold for all t. For brevity, we denote the sum of the mass and input distributions by $\nu_t := \mu_t + u_t$. Since u_t satisfies Assumption 2.1, if statement (i) holds at any t, then $\nu_t \ll \mathcal{L}^1$ is finite and $\operatorname{supp}\nu_t$ is a closed bounded interval. Hence, ν_t's density function $\varrho_t(x) \geq 0$ exists and satisfies $\varrho_t(x) \in [\varrho_{\min}(t), \varrho_{\max}(t)]$ for all $x \in \operatorname{supp}\nu_t$ with $0 < \varrho_{\min}(t) \leq \varrho_{\max}(t) < \infty$.

Regarding part (ii), for any $x, y \in \operatorname{supp}\mu_t$ and $x < y$, since $x \pm r$ or $y \pm r$ may not belong to $\operatorname{supp}\nu_t$,

$$\gamma_t(x) = \frac{\int_a^b z\varrho_t(z)dz}{\int_a^b \varrho_t(z)dz}, \quad \gamma_t(y) = \frac{\int_p^q z\varrho_t(z)dz}{\int_p^q \varrho_t(z)dz}, \tag{3.1}$$

where $[a, b] = [x - r, x + r] \cap \operatorname{supp}\nu_t$ and $[p, q] = [y - r, y + r] \cap \operatorname{supp}\nu_t$. Equivalently,

$$\gamma_t(x) = \frac{\int_a^p z\varrho_t(z)dz + \int_p^b z\varrho_t(z)dz}{\int_a^p \varrho_t(z)dz + \int_p^b \varrho_t(z)dz} =: \frac{\hat{S}_1 + \hat{S}_2}{S_1 + S_2}, \tag{3.2}$$

$$\gamma_t(y) = \frac{\int_p^b z\varrho_t(z)dz + \int_b^q z\varrho_t(z)dz}{\int_p^b \varrho_t(z)dz + \int_b^q \varrho_t(z)dz} =: \frac{\hat{S}_2 + \hat{S}_3}{S_2 + S_3}, \tag{3.3}$$

where $\hat{S}_1$ and S_1 denote $\int_a^p z\varrho_t(z)dz$ and $\int_a^p \varrho_t(z)dz$, respectively, and $\hat{S}_{2,3}$ and $S_{2,3}$ are defined similarly. It follows from the properties of ν_t that Lemma 2.1 holds for ν_t over any closed interval in $\mathbb{R}$, and considering the integration intervals of $\hat{S}_i$'s and S_i's, for nonzero S_i's we have

$$\frac{\hat{S}_1}{S_1} < \frac{\hat{S}_2}{S_2} < \frac{\hat{S}_3}{S_3} \quad \Rightarrow \quad \hat{S}_1 S_2 < \hat{S}_2 S_1, \ \ \hat{S}_2 S_3 < \hat{S}_3 S_2, \ \text{ and } \ \hat{S}_1 S_3 < \hat{S}_3 S_1.$$

Notice that based on assumption $|\operatorname{supp}\mu_t| > 2r$, at least one of the S_1 or S_3 should be nonzero; moreover, since $\operatorname{supp}\nu_t$ is a closed interval, the terms $S_1 + S_2$ and $S_1 + S_3$ are nonzero. Consequently, only one term out of the three terms S_1, S_2, and S_3 can be zero, and the following inequality always holds:

$$\hat{S}_1 S_2 + \hat{S}_1 S_3 + \hat{S}_2 S_2 + \hat{S}_2 S_3 < \hat{S}_2 S_1 + \hat{S}_2 S_2 + \hat{S}_3 S_1 + \hat{S}_3 S_2$$
$$\Rightarrow \frac{\hat{S}_1 + \hat{S}_2}{S_1 + S_2} < \frac{\hat{S}_2 + \hat{S}_3}{S_2 + S_3} \quad \Rightarrow \quad \gamma_t(x) < \gamma_t(y).$$

Regarding part (iii), the bi-Lipschitz property of the flow map $\gamma_t(x)$ asserts that for any $x, y \in \operatorname{supp}\mu_t$ (2.3) holds for some $L_t \geq 1$. Assume that $x < y$, and, according to part (ii), $\gamma_t(x) < \gamma_t(y)$. Then, two different cases are possible:

(1) $y - x \geq 2r$; hence,

$$\gamma_t(y) - \gamma_t(x) < y - x + 2r \leq 2(y - x),$$

and it follows from Lemma 2.1 that

$$\gamma_t(y) - \gamma_t(x) > \frac{(b-a) + (q-p)}{1 + \sqrt{\varrho_{\max}(t)/\varrho_{\min}(t)}},$$

where the flow maps are given by (3.1). Since $y - x \leq |\operatorname{supp}\mu_t|$,

$$\frac{(b-a) + (q-p)}{1 + \sqrt{\varrho_{\max}(t)/\varrho_{\min}(t)}} = \frac{(b-a+q-p)(y-x)}{(1 + \sqrt{\varrho_{\max}(t)/\varrho_{\min}(t)})(y-x)}$$
$$\geq \frac{(b-a+q-p)(y-x)}{|\operatorname{supp}\mu_t|(1 + \sqrt{\varrho_{\max}(t)/\varrho_{\min}(t)})}.$$

Finally,

$$L_t = \max\left\{2, \frac{|\operatorname{supp}\mu_t|(1 + \sqrt{\varrho_{\max}(t)/\varrho_{\min}(t)})}{b-a+q-p}\right\}.$$

Since $b - a + q - p \leq |\operatorname{supp}\mu_t| + 2r$, $1 + \sqrt{\varrho_{\max}(t)/\varrho_{\min}(t)} \geq 2$, and $|\operatorname{supp}\mu_t| > 2r$,

$$\frac{|\operatorname{supp}\mu_t|(1 + \sqrt{\varrho_{\max}(t)/\varrho_{\min}(t)})}{b-a+q-p} \geq \frac{2|\operatorname{supp}\mu_t|}{|\operatorname{supp}\mu_t| + 2r} \geq 1,$$

which confirms that $L_t \geq 1$.

(2) $y - x < 2r$; hence, following (3.2) and (3.3), we have

$$\gamma_t(y) - \gamma_t(x) = \frac{\hat{S}_2 + \hat{S}_3}{S_2 + S_3} - \frac{\hat{S}_1 + \hat{S}_2}{S_1 + S_2} = \frac{\hat{S}_2 S_1 + \hat{S}_3 S_1 + \hat{S}_3 S_2 - \hat{S}_1 S_2 - \hat{S}_1 S_3 - \hat{S}_2 S_3}{(S_1 + S_2)(S_2 + S_3)}.$$

Based on statement (i), the following inequalities can be derived:

$$\begin{aligned}
&S_1 < (p-a)\varrho_{\max}(t) \leq (y-x)\varrho_{\max}(t), && \hat{S}_1 < p(p-a)\varrho_{\max}(t) \leq q(y-x)\varrho_{\max}(t),\\
&S_3 < (q-b)\varrho_{\max}(t) \leq (y-x)\varrho_{\max}(t), && \hat{S}_3 < q(q-b)\varrho_{\max}(t) \leq q(y-x)\varrho_{\max}(t),\\
&S_2 < (b-p)\varrho_{\max}(t) \leq 2r\varrho_{\max}(t), && \hat{S}_2 < b(b-p)\varrho_{\max}(t) \leq 2rq\varrho_{\max}(t),\\
&(S_1 + S_2)(S_2 + S_3) > r^2\varrho_{\min}(t)^2.
\end{aligned}$$

Consequently,

$$\gamma_t(y) - \gamma_t(x) < \frac{2r|q|\varrho_{\max}(t)^2(y-x) + |q|\varrho_{\max}(t)^2(y-x)^2 + 2r|q|\varrho_{\max}(t)^2(y-x)}{r^2\varrho_{\min}(t)^2}.$$

Again, since $y - x < 2r$ and $|q| \le \max\{|x_{\max}(t)|, |x_{\min}(t)|\}$,

$$\gamma_t(y) - \gamma_t(x) < \frac{6r\max\{|x_{\max}(t)|, |x_{\min}(t)|\}\varrho_{\max}(t)^2}{r^2\varrho_{\min}(t)^2}(y-x) =: L_1(y-x).$$

It follows from $|\operatorname{supp}\mu_t| > 2r$ that $\max\{|x_{\max}(t)|, |x_{\min}(t)|\} \ge r$, and thus $L_1 > 1$. As stated above, either S_1 or S_3 is nonzero. Without loss of generality, assume that S_1 is nonzero, and hence $p - a = y - x$. It follows from

$$\frac{\hat{S}_3 + \hat{S}_2}{S_3 + S_2} \ge \frac{\hat{S}_2}{S_2}$$

that

$$\gamma_t(y) - \gamma_t(x) \ge \frac{\hat{S}_2}{S_2} - \frac{\hat{S}_1 + \hat{S}_2}{S_1 + S_2} =: c(x_2 - x_1), \tag{3.4}$$

where

$$x_1 = \frac{\hat{S}_1}{S_1}, \quad x_2 = \frac{\hat{S}_2}{S_2}, \quad \text{and} \quad c = \frac{S_1}{S_1 + S_2} \ge \frac{(p-a)\varrho_{\min}(t)}{(b-p)\varrho_{\max}(t) + (p-a)\varrho_{\max}(t)}$$
$$\Rightarrow \gamma_t(y) - \gamma_t(x) \ge \frac{(p-a)\varrho_{\min}(t)}{(b-a)\varrho_{\max}(t)}(x_2 - x_1) > \frac{(y-x)\varrho_{\min}(t)}{2r\varrho_{\max}(t)}(x_2 - x_1).$$

By Lemma 2.1,

$$x_2 - x_1 > \frac{b - p + p - a}{\sqrt{\varrho_{\max}(t)/\varrho_{\min}(t)} + 1} \ge \frac{r}{\sqrt{\varrho_{\max}(t)/\varrho_{\min}(t)} + 1}.$$

Therefore,

$$\gamma_t(y) - \gamma_t(x) > \frac{\varrho_{\min}(t)}{2\varrho_{\max}(t)(\sqrt{\varrho_{\max}(t)/\varrho_{\min}(t)} + 1)}(y-x) =: \frac{1}{L_2}(y-x).$$

Clearly $L_2 \ge 1$; therefore, $L_t = \max\{L_1, L_2\}$.

Regarding part (i), we now prove that if statements (i), (ii), and (iii) hold at time t, then statement (i) holds at $t+1$. First, we prove that the flow map

$$\gamma_t(x) = \frac{\int_{x-r}^{x+r} z\varrho_t(z)dz}{\int_{x-r}^{x+r} \varrho_t(z)dz} =: \frac{f(x)}{g(x)}$$

is continuous. Knowing that if two functions f and g are continuous and $g \neq 0$, then the quotient f/g is also continuous, we show the continuity of the function $f(x)$ at all points

$c \in \operatorname{supp} \mu_t$, and the proof of the continuity of the denominator is similar. For all $x \in \operatorname{supp} \mu_t$, $g(x) > 0$, and

$$\begin{aligned}\lim_{x\to c} f(x) = \lim_{x\to c}\int_{x-r}^{x+r} z d\varrho(z) = \lim_{\epsilon\to 0}\int_{c-\epsilon-r}^{c+\epsilon+r} z d\varrho(z) \\ = \lim_{\epsilon\to 0}\left(\int_{c-\epsilon-r}^{c-r} z d\varrho(z) + \int_{c+r}^{c+\epsilon+r} z d\varrho(z)\right) + \int_{c-r}^{c+r} z d\varrho(z) \\ = \lim_{\epsilon\to 0}\big(\epsilon(c-r)\varrho(c-r) + \epsilon(c+r)\varrho(c+r)\big) + f(c).\end{aligned}$$

Due to the finiteness and absolute continuity of ν_t, $f(c)$ exists for all $c \in \operatorname{supp}\mu_t$ and the limit in the right-hand side converges to zero; hence, $\lim_{x\to c} f(x) = f(c)$. We have shown that $\gamma_t(x)$ is strictly monotone and continuous with respect to x; therefore, this map is also invertible. Second, we prove the absolute continuity of μ_{t+1}. It is shown that γ_t has the following properties: (1) Since any continuous function defined on Borel sets is a Borel measurable function, $\gamma_t^{-1}(E)$ is Borel measurable for any Borel set $E \in \mathbb{R}$. (2) The bi-Lipschitz map γ_t satisfies

$$\mathcal{L}(\gamma_t^{-1}(E)) \leq C_t \mathcal{L}(E)$$

for some constant $C_t \in \mathbb{R}_{>0}$. According to Theorem 2 in [26], if the flow map γ_t satisfies the above two properties and $\mu_t \ll \mathcal{L}^1$, then $\mu_{t+1} \ll \mathcal{L}^1$. Third, a continuous function maps a compact set to another compact set; hence, γ_t maps the closed bounded interval $\operatorname{supp}\mu_t$ to another closed bounded interval $\operatorname{supp}\mu_{t+1}$. Fourth, we establish bounds on μ_{t+1}'s density function. For any $x, y \in \operatorname{supp}\mu_{t+1}$ and $x < y$, (2.2) gives

$$\int_x^y \rho_{t+1}(z)dz = \int_{\gamma_t^{-1}(x)}^{\gamma_t^{-1}(y)} \rho_t(z)dz,$$

where $\rho_t(z)$ is μ_t's density function, and, in view of condition (i), $0 < \rho_{\min}(t) \leq \rho_t(z) \leq \rho_{\max}(t) < \infty$ over $\operatorname{supp}\mu_t$. Therefore,

$$(\gamma_t^{-1}(y) - \gamma_t^{-1}(x))\rho_{\min}(t) \leq \int_{\gamma_t^{-1}(x)}^{\gamma_t^{-1}(y)} \rho_t(z)dz \leq (\gamma_t^{-1}(y) - \gamma_t^{-1}(x))\rho_{\max}(t).$$

Provided that γ_t is bi-Lipschitz, for any $x, y \in \operatorname{supp}\mu_t$, there exists $L_t \geq 1$ such that

$$\begin{aligned}\frac{1}{L_t}(y-x)\rho_{\min}(t) \leq \int_{\gamma_t^{-1}(x)}^{\gamma_t^{-1}(y)} \rho_t(z)dz \leq L_t(y-x)\rho_{\max}(t) \\ \Rightarrow \frac{1}{L_t}(y-x)\rho_{\min}(t) \leq \int_x^y \rho_{t+1}(z)dz \leq L_t(y-x)\rho_{\max}(t).\end{aligned}$$

The limit of the above inequality as y converges to x gives

$$\begin{aligned}\frac{1}{L_t}(y-x)\rho_{\min}(t) \leq (y-x)\rho_{t+1}(x) \leq L_t(y-x)\rho_{\max}(t) \\ \Rightarrow \frac{1}{L_t}\rho_{\min}(t) \leq \rho_{t+1}(x) \leq L_t\rho_{\max}(t) \quad \forall x \in \operatorname{supp}\mu_{t+1}.\end{aligned}$$

Finally, since $\operatorname{supp}\mu_t$ is bounded for all $t \geq 0$, we have

$$\mu_{t+1}(\operatorname{supp}\mu_{t+1}) = \mu_t(\gamma_t^{-1}(\operatorname{supp}\mu_{t+1})) = \mu_t(\operatorname{supp}\mu_t).$$

Therefore, if μ_t is finite, then μ_{t+1} is finite. ■

Lemma 3.2 (sufficient condition for consensus). *Assume that an Eulerian HK system with input satisfies Assumption 2.2, $\mu_0, u_0 \ll \mathcal{L}^1$ are finite measures, and $\operatorname{supp}\mu_0$ is closed and bounded. If μ_0 and u_t for all $t \geq 0$ are distributed symmetrically around the center of $\operatorname{supp}\mu_0$, and $|\operatorname{supp}\mu_0| \leq 2r$, then the mass distribution reaches an opinion consensus in finite time.*

Proof. Let us denote $\mu_t + u_t$ by ν_t. Owing to the absolute continuity of μ_0 and u_0 and according to Lemma 2.1, $x_{\min}(1) > x_{\min}(0)$ and $x_{\max}(1) < x_{\max}(0)$. Since ν_0 and u_t are symmetrically distributed and the confidence bounds are homogeneous for all opinions, the distribution ν_t remains symmetric around the center of $\operatorname{supp}\nu_t$ for all $t \geq 0$. Hence, the value $(x_{\min}(t) + x_{\max}(t))/2$ for all $t \geq 0$ is a constant, denoted by x_{mid}. In view of the absolute continuity of ν_0, $\gamma_0(x)$ is a continuous function of x, and thus, for any $\zeta \in \mathbb{R}_{>0}$, there exists $\epsilon \in \mathbb{R}_{>0}$ such that

$$\gamma_0(x_{\text{mid}} + \epsilon) - x_{\text{mid}} < \zeta. \tag{3.5}$$

If we let $\zeta = x_{\max}(0) - x_{\max}(1) = x_{\min}(1) - x_{\min}(0)$, then (3.5) gives $\mu_1([x_{\text{mid}} - \zeta, x_{\text{mid}} + \zeta]) \geq \mu_0([x_{\text{mid}} - \epsilon, x_{\text{mid}} + \epsilon]) > 0$. Moreover, at time $t = 1$, the population over $[x_{\text{mid}} - \zeta, x_{\text{mid}} + \zeta]$ considers the total population's opinion in its opinion update and thus reaches consensus at x_{mid} in the next iteration. Consequently, at $t = 2$, there exists an atom at x_{mid} whose measure is denoted by ν_{mid}. Since $\gamma_t(x_{\text{mid}}) = x_{\text{mid}}$, the measure of the atom at x_{mid} is greater than or equal to ν_{mid} for all $t \geq 2$. Now, for all $t \geq 2$, we compute a strictly positive lower bound for $x_{\min}(t+1) - x_{\min}(t)$, which is equal to the lower bound on $x_{\max}(t) - x_{\max}(t+1)$ and proves that $|\operatorname{supp}\nu_t|$ is strictly decreasing and converges to zero. Since $|\operatorname{supp}\nu_t| \leq |\operatorname{supp}\nu_0| \leq 2r$ for all $t \geq 0$, the intervals $[x_{\min}(t), x_{\min}(t) + r]$ and $[x_{\max}(t) - r, x_{\max}(t)]$ contain the central point x_{mid}. For all $x \in [x_{\min}(t), x_{\text{mid}})$,

$$\gamma_t(x) \geq \frac{x_{\min}(t)\hat{\nu}_t + x_{\text{mid}}\nu_{\text{mid}}}{\hat{\nu}_t + \nu_{\text{mid}}},$$

where $\hat{\nu}_t := \int_{[x_{\min}(t), x_{\min}(t)+r]} d\nu_t(x) - \nu_{\text{mid}}$, and since $\nu_t(\mathbb{R})$ is finite, we denote $\hat{\nu}_t$'s upper bound for all $t \geq 0$ such that $x_{\min}(t) < x_{\text{mid}}$ by $\hat{\nu}_{\max} \in \mathbb{R}_{>0}$. Therefore,

$$x_{\min}(t+1) \geq x_{\min}(t) + \frac{(x_{\text{mid}} - x_{\min}(t))\nu_{\text{mid}}}{\hat{\nu}_{\max} + \nu_{\text{mid}}},$$

which implies that the lower bound on $x_{\min}(t+1) - x_{\min}(t)$ is $(x_{\text{mid}} - x_{\min}(t))$ multiplied by a constant. Consequently, there exists time $\tau \geq 0$ such that $x_{\text{mid}} - x_{\min}(\tau) < r - (x_{\max}(\tau) - x_{\min}(\tau))/2$, and thus the mass distribution reaches a consensus at $\tau + 1$. ■

4. Convergence behavior.

In this section, we first introduce a lemma which guarantees that the length of the mass distribution's support strictly monotonically decreases as time progresses. Next, employing this result and the system properties introduced in section 3, the convergence of the mass distribution in an Eulerian HK system without input is established.

Lemma 4.1. *If an Eulerian HK system with input satisfies Assumptions 2.1 and 2.2, then, for all $t \geq 0$ such that $|\operatorname{supp}\mu_t| > 2r$,*

(i) $\operatorname{supp}\mu_t$ *strictly contains* $\operatorname{supp}\mu_{t+1}$, *and*

(ii) $x_{\min}(t+1) = \gamma_t(x_{\min}(t))$ *and* $x_{\max}(t+1) = \gamma_t(x_{\max}(t))$.

Proof. This system satisfies the sufficient conditions of Theorem 3.1, and part (i) of Theorem 3.1 shows that $\operatorname{supp}\mu_t$ is equal to the closed bounded interval $[x_{\min}(t), x_{\max}(t)]$ for all $t \geq 0$. Hence, statement (i) can be concluded from

$$x_{\min}(t) < x_{\min}(t+1) < x_{\max}(t+1) < x_{\max}(t). \tag{4.1}$$

Next, we prove the lower bound in inequality (4.1), and the proof to the upper bound is similar. For all t, based on Assumption 2.2, the support of measure $\nu_t = \mu_t + u_t$ is equal to $\operatorname{supp}\mu_t$. Therefore, the density function of ν_t is equal to zero below $x_{\min}(t)$ and strictly greater than zero above $x_{\min}(t)$, and hence $\gamma_t(x_{\min}(t)) > x_{\min}(t)$. According to Theorem 3.1 part (ii), for all $y \in \operatorname{supp}\mu_t$ and $y > x_{\min}(t)$, $\gamma_t(y) > \gamma_t(x_{\min}(t))$. Therefore, $\gamma_t(x_{\min}(t))$ is the smallest opinion in the set $\operatorname{supp}\mu_{t+1}$, i.e., $\gamma_t(x_{\min}(t)) = x_{\min}(t+1)$, and thus $x_{\min}(t+1) > x_{\min}(t)$. ∎

Notice that if $\operatorname{supp} u_t$ is not contained in $\operatorname{supp}\mu_t$, then $\operatorname{supp}\mu_{t+1}$ is not necessarily contained in $\operatorname{supp}\mu_t$. Lemma 4.1 also holds for an Eulerian HK system without input, that is, $u_t = 0$ for all $t \geq 0$.

Theorem 4.2. *Consider an Eulerian HK system with no input, confidence bound r, and initial condition such that $\mu_0 \ll \mathcal{L}^1$ is finite and $\operatorname{supp}\mu_0$ is a closed bounded interval. If $|\operatorname{supp}\mu_t| > 2r$ for all $t \geq 0$, then μ_t converges in the weak-star topology to an atomic measure, whose atoms are separated by a distance greater than r.*

Proof. This system satisfies the conditions of Theorem 3.1 and Lemma 4.1. Therefore, $\mu_t \ll \mathcal{L}^1$, $\operatorname{supp}\mu_t$ is a closed bounded interval, and $\operatorname{supp}\mu_t \subset \operatorname{supp}\mu_{t-1}$. Since $x_{\min}(t)$ is a strictly increasing function of time and $\operatorname{supp}\mu_t$ is a subset of $\operatorname{supp}\mu_0$, there exists an opinion x_1 in the interior of $\operatorname{supp}\mu_t$ such that $\lim_{t\to\infty} x_{\min}(t) = x_1$. Thus, there exists τ such that $x_1 - x_{\min}(t) < r$ for all $t \geq \tau$, and in the remainder of this proof t is assumed to be larger than τ. First, we prove that the mass distribution over interval $(x_1, x_1 + r)$ converges to zero. Let us denote the intervals $[x_{\min}(t), x_1]$ and $(x_1, x_{\min}(t)+r]$ by I_t and $\hat{I}_t$, respectively, and the density function of μ_t by ρ_t. Then, we define the following opinion average:

$$y_t(\hat{I}_t) := \frac{\int_{x_1}^{x_{\min}(t)+r} z\rho_t(z)dz}{\int_{x_1}^{x_{\min}(t)+r} \rho_t(z)dz}.$$

By Lemma 4.1, $x_{\min}(t+1) = \gamma_t(x_{\min}(t))$; hence

$$\lim_{t\to\infty} x_{\min}(t+1) = \lim_{t\to\infty} \frac{\int_{x_{\min}(t)}^{x_1} z\rho_t(z)dz + y_t(\hat{I}_t)\mu_t(\hat{I}_t)}{\int_{x_{\min}(t)}^{x_1} \rho_t(z)dz + \mu_t(\hat{I}_t)}.$$

Moreover, since $x_{\min}(t)$ converges to x_1 as time goes to infinity,

$$\lim_{t\to\infty} \int_{x_{\min}(t)}^{x_1} z\rho_t(z)dz = x_1 \lim_{t\to\infty} \mu_t(I_t).$$

Hence,

$$\lim_{t\to\infty} x_{\min}(t+1) - x_1 = \lim_{t\to\infty} \frac{x_1\mu_t(I_t) + y_t(\hat{I}_t)\mu_t(\hat{I}_t)}{\mu_t(I_t) + \mu_t(\hat{I}_t)} - x_1 = \lim_{t\to\infty} \frac{(y_t(\hat{I}_t) - x_1)\mu_t(\hat{I}_t)}{\mu_t(I_t) + \mu_t(\hat{I}_t)}. \tag{4.2}$$

Therefore, the limit $\lim_{t\to\infty} x_{\min}(t+1) - x_1 = 0$ results in

$$\lim_{t\to\infty} \frac{y_t(\hat{I}_t) - x_1}{\frac{\mu_t(I_t)}{\mu_t(\hat{I}_t)} + 1} = 0.$$

For the limit above to hold, it is necessary that at least one of the following cases hold true:

(1) $\mu_t(\hat{I}_t)$ converges to zero. Hence, for any $\epsilon, \delta \in \mathbb{R}_{>0}$, there exists time $T \geq 0$ such that for all $t \geq T$, $\mu_t((x_1, x_1 + r - \delta]) < \epsilon$. Therefore, $\mu_t((x_1, x_1 + r))$ converges to zero.

(2) $y_t(\hat{I}_t)$ converges to x_1. Then, due to the absolute continuity of μ_t, $y_t([x_1, x_{\min}(t) + r])$ converges to x_1. Hence, according to Lemma 2.2, the mass distribution over $[x_1, x_{\min}(t) + r]$ converges to a weighted Dirac delta distribution centered at x_1. Therefore, $\mu_t((x_1, x_{\min}(t)+r])$ converges to zero and, similar to above, we conclude that $\mu_t((x_1, x_1 + r))$ converges to zero.

Second, since $\mu_t(\operatorname{supp} \mu_t)$ is lower bounded, two cases are possible: (i) μ_t converges to a single atom at x_1, which proves our theorem; and (ii) there exists opinion $\hat{x}_2 \in [x_1 + r, x_{\max}(t)]$ such that $\rho_t(\hat{x}_2) > \rho_{\min}$ for some $\rho_{\min} \in \mathbb{R}_{>0}$ and all $t \geq 0$. This is due to the fact that $\operatorname{supp} \mu_t$ is a closed bounded interval and over this interval the density ρ_t is strictly greater than zero. Since $\mu_t(\operatorname{supp} \mu_t)$ is lower bounded and μ_t is not converging to a single atom, there exists such an opinion $\hat{x}_2$ whose density does not converge to zero and thus is lower bounded. Denote the infimum of such $\hat{x}_2$'s by x_2. Therefore, for any $x \in [x_1 + r, x_2)$, $\rho_t(x)$ converges to zero, and thus $\mu_t([x_1 + r, x_2))$ converges to zero. According to the first part of this proof, $\mu_t((x_1, x_1 + r))$ also converges to zero; hence, owing to $x_2 \geq x_1 + r$, $\mu_t([x_2 - r, x_2))$ converges to zero as time goes to infinity. Let us denote the opinion intervals $[x_2 - r, x_2)$ and $[x_2, x_2 + r]$ by J_1 and J_2, and their opinion averages by $y_t(J_1)$ and $y_t(J_2)$, respectively. Then,

$$\gamma_t(x_2) = \frac{y_t(J_1)\mu_t(J_1) + y_t(J_2)\mu_t(J_2)}{\mu_t(J_1) + \mu_t(J_2)}.$$

Next, we prove that either $y_t(J_2)$ converges to x_2 or $\mu_t(J_2)$ converges to zero. It is known from Lemma 4.1 part (i) that $y_t(J_2) - x_2 > 0$ and since $x_2 \in \operatorname{supp} \mu_t$, $\mu_t(J_2) > 0$. Therefore, if $y_t(J_2) - x_2$ and $\mu_t(J_2)$ do not converge to zero, they are lower bounded for all times. By contradiction, assume that there exist $\kappa \in \mathbb{R}_{>0}$ and $\mu_{\min} \in \mathbb{R}_{>0}$ such that $y_t(J_2) - x_2 > \kappa$ and $\mu_t(J_2) > \mu_{\min}$ for all $t \geq 0$. As stated above, for any $\epsilon \in \mathbb{R}_{>0}$, there exists $T \geq 0$ such that $\mu_t(J_1) < \epsilon$ for all $t \geq T$. Knowing that $y_t(J_1) - x_2 > -r$, (2.6) implies that for all $t \geq T$

$$\gamma_t(x_2) - x_2 = \frac{(y_t(J_1) - x_2)\mu_t(J_1) + (y_t(J_2) - x_2)\mu_t(J_2)}{\mu_t(J_1) + \mu_t(J_2)} > \frac{-r\epsilon + \kappa\mu_{\min}}{\epsilon + \mu_{\min}}.$$

Consider $\epsilon_1, \delta_1 \in \mathbb{R}_{>0}$ such that $\epsilon_1 < \kappa\mu_{\min}/r$ and

$$\delta_1 = \frac{-r\epsilon_1 + \kappa\mu_{\min}}{\epsilon_1 + \mu_{\min}}.$$

It follows from $\rho_t(x_2) > \rho_{\min}$ for infinite time steps and the absolute continuity of μ_t that for any $\delta \in \mathbb{R}_{>0}$, there exists $\epsilon_2 \in \mathbb{R}_{>0}$ such that $\mu_t(\mathcal{B}_\delta(x_2)) > \epsilon_2$ for infinite $t \geq 0$, where $\mathcal{B}_\delta(x)$ is an open ball centered at x with radius δ. Now, let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and $\delta = \delta_1$; then again by (2.6) for an infinite number of time steps $t \geq T$,

$$\delta = \frac{-r\epsilon_1 + \kappa\mu_{\min}}{\epsilon_1 + \mu_{\min}} \leq \frac{-r\epsilon + \kappa\mu_{\min}}{\epsilon + \mu_{\min}} < \gamma_t(x_2) - x_2.$$

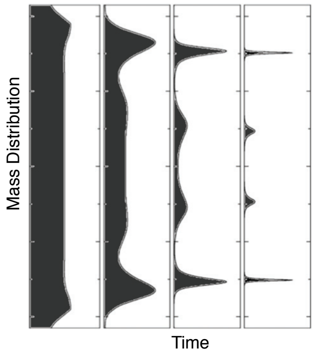

Figure 2. *A schematic illustration of convergence of an Eulerian HK system with no input in the weak-star topology to an atomic measure, whose atoms are separated by a distance greater than r.*

Therefore, for any $x \in \mathcal{B}_\delta(x_2)$, $x < \gamma_t(x_2)$, and thus according to Theorem 3.1 part (ii), $\gamma_t^{-1}(\mathcal{B}_\delta(x_2)) \in J_1$ for infinite t's. Based on (2.2),

$$\mu_{t+1}(\mathcal{B}_\delta(x_2)) = \mu_t(\gamma_t^{-1}(\mathcal{B}_\delta(x_2))) < \mu_t(J_1) < \epsilon,$$

which contradicts the assumption that $\mu_t(\mathcal{B}_\delta(x_2)) > \epsilon_2 \geq \epsilon$ for all $t \geq 0$. Therefore, it is true that either $y_t(J_2)$ converges to x_2 or $\mu_t(J_2)$ converges to zero. In the former case, Lemma 2.2 shows that $\mu_t([x_2, x_2 + r])$ converges to a weighted Dirac delta distribution centered at x_2, and thus it can be concluded from both cases that $\mu_t((x_2, x_2 + r])$ converges to zero.

Third, we repeat the second part of this proof for the opinion interval $[x_2 + r, x_{\max}(t)]$ and so on.

Finally, for every bounded and continuous test function η,

$$\lim_{t\to\infty} \int_{\mathbb{R}} \eta(z)\mu_t(dz) = \eta(x_1)m_1 + \eta(x_2)m_2 + \eta(x_3)m_3 + \cdots =: \int_{\mathbb{R}} \eta(z)\mu_\infty(dz),$$

where the measures $\mu_\infty([x_{\min}(\infty), x_1 + r))$, $\mu_\infty([x_1 + r, x_2 + r))$, $\mu_\infty([x_2 + r, x_3 + r))$, $\ldots$ are denoted by m_1, m_2, m_3, $\ldots$, respectively. Hence, μ_t converges in the weak-star topology to an atomic measure μ_∞, whose atoms, $\{m_i : i = 1, 2, 3, \ldots\}$, are far apart with at least distance r; see Figure 2. ■

Following our theorem on the convergence of the mass distribution to separate atoms, the next lemma establishes the clustering behavior in one iteration in our Eulerian HK system with input and the idea of "rich gets richer and poor gets poorer." Roughly speaking, the lemma states that for a population distributed over some small opinion interval $[a, b]$, if the density of the population and the exogenous input over the opinion interval $[a - r, b + r]$ is higher than the density of the population just outside this interval, then the population over $[a, b]$ concentrates in one iteration.

In the remainder of this paper, we denote $\gamma_t(x) - x$ by $\Delta_t(x)$ for brevity.

Lemma 4.3. *Consider an Eulerian HK system with input that satisfies Assumption 2.1. Denote the sum of mass distribution and input ($\mu_t + u_t$) by measure ν_t and the density functions of μ_t and ν_t by $\rho_t : \operatorname{supp} \mu_t \to \mathbb{R}_{\geq 0}$ and $\varrho_t : \operatorname{supp} \nu_t \to \mathbb{R}_{\geq 0}$, respectively. Then for any $x \in \operatorname{supp} \mu_t$ such that $\rho_t(x) > 0$ and ρ_t is continuous at x, the following statements hold:*

(i) ρ_{t+1} *is continuous at* $\gamma_t(x)$ *and* $\rho_{t+1}\big(\gamma_t(x)\big) > 0$*;*
(ii) $\partial\Delta_t(x)/\partial x = \rho_t(x)/\rho_{t+1}\big(\gamma_t(x)\big) - 1$*; and*
(iii) *if*

$$\max\{\varrho_t(x+r), \varrho_t(x-r)\} < \frac{\nu_t([x-r, x+r])}{2r} \quad or \tag{4.3}$$

$$\min\{\varrho_t(x+r), \varrho_t(x-r)\} > \frac{\nu_t([x-r, x+r])}{2r}, \tag{4.4}$$

then $\rho_{t+1}\big(\gamma_t(x)\big) > \rho_t(x)$ *or* $\rho_{t+1}\big(\gamma_t(x)\big) < \rho_t(x)$*, respectively.*

Proof. Notice that the underlying system satisfies the conditions of Theorem 3.1, and hence μ_t and ν_t are absolutely continuous for all $t \geq 0$ and $\operatorname{supp}\mu_t$ is a closed bounded interval.

Regarding part (i), first denote a ball of infinitesimal radius $\epsilon \in \mathbb{R}_{>0}$ centered at any opinion x by $\mathcal{B}_\epsilon(x)$; then the absolute continuity of μ_t and μ_{t+1} together with (2.2) result in

$$\begin{aligned}\rho_{t+1}(\gamma_t(x)) = \lim_{\epsilon\to 0}\frac{\mu_{t+1}(\mathcal{B}_\epsilon(\gamma_t(x)))}{2\epsilon} &= \lim_{\epsilon\to 0}\frac{\mu_t\big(\gamma_t^{-1}(\mathcal{B}_\epsilon(\gamma_t(x)))\big)}{2\epsilon}\\ &= \rho_t(x)\lim_{\epsilon\to 0}\frac{\gamma_t^{-1}(\gamma_t(x)+\epsilon) - \gamma_t^{-1}(\gamma_t(x)-\epsilon)}{2\epsilon},\end{aligned}$$

where the right-hand side is a result of continuity of γ_t, γ_t^{-1}, and ρ_t at x. Furthermore, the bi-Lipschitz continuity of γ_t together with the equality above implies that ρ_{t+1} is continuous at $\gamma_t(x)$. Hence, we can write

$$\rho_t(x) = \lim_{\epsilon\to 0}\frac{\mu_t(\mathcal{B}_\epsilon(x))}{2\epsilon} = \lim_{\epsilon\to 0}\frac{\mu_{t+1}\big(\gamma_t(\mathcal{B}_\epsilon(x))\big)}{2\epsilon} = \rho_{t+1}\big(\gamma_t(x)\big)\lim_{\epsilon\to 0}\frac{\gamma_t(x+\epsilon)-\gamma_t(x-\epsilon)}{2\epsilon}. \tag{4.5}$$

According to Theorem 3.1 parts (ii) and (iii), $2L_t\epsilon > \gamma_t(x+\epsilon) - \gamma_t(x-\epsilon) > 2\epsilon/L_t$ for some $L_t \geq 1$. Therefore,

$$\rho_t(x)/L_t < \rho_{t+1}\big(\gamma_t(x)\big) < L_t\rho_t(x),$$

which proves that $\rho_{t+1}\big(\gamma_t(x)\big) > 0$.

Regarding part (ii), we have

$$\frac{\partial\Delta_t(x)}{\partial x} = \lim_{\epsilon\to 0}\frac{\Delta_t(x+\epsilon)-\Delta_t(x-\epsilon)}{2\epsilon} = \lim_{\epsilon\to 0}\frac{\gamma_t(x+\epsilon)-\gamma_t(x-\epsilon)}{2\epsilon} - 1.$$

Employing (4.5) in the right-hand side results in the claimed statement.

Regarding part (iii), we show that if the inequality (4.3) holds true, then $\rho_{t+1}\big(\gamma_t(x)\big) > \rho_t(x)$, and the proof of the second inequality is similar. Consider

$$\begin{aligned}\lim_{\epsilon\to 0}\gamma_t(x+\epsilon) &= \lim_{\epsilon\to 0}\frac{(x+r)S_1 + z_tS_2}{S_1+S_2},\\ \lim_{\epsilon\to 0}\gamma_t(x-\epsilon) &= \lim_{\epsilon\to 0}\frac{(x-r)S_3 + z_tS_2}{S_3+S_2},\end{aligned}$$

where $S_{1,2,3}$ denote $\nu_t(\mathcal{B}_\epsilon(x+r))$, $\nu_t([x-r+\epsilon, x+r-\epsilon])$, and $\nu_t(\mathcal{B}_\epsilon(x-r))$, respectively, and z_t denotes the opinion average over the interval $[x-r+\epsilon, x+r-\epsilon]$. Knowing that $\lim_{\epsilon\to 0}\nu_t(\mathcal{B}_\epsilon(x+r))/2\epsilon = \varrho_t(x+r)$ and $\lim_{\epsilon\to 0}\nu_t(\mathcal{B}_\epsilon(x-r))/2\epsilon = \varrho_t(x-r)$,

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{\gamma_t(x+\epsilon)-\gamma_t(x-\epsilon)}{2\epsilon} &= \lim_{\epsilon\to 0}\frac{\gamma_t(x+\epsilon)-x-\gamma_t(x-\epsilon)+x}{2\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{2rS_1S_3/\epsilon+\varrho_t(x+r)S_2(x-z_t+r)+\varrho_t(x-r)S_2(z_t-x+r)}{(S_1+S_2)(S_3+S_2)}.\end{aligned}$$

Since $|z_t-x|<r$, both $x-z_t+r$ and z_t-x+r are positive, and, according to the inequality (4.3), $\max\{\varrho_t(x+r),\varrho_t(x-r)\} < \lim_{\epsilon\to 0} S_2/2r$. Therefore,

$$\begin{aligned}&\lim_{\epsilon\to 0}\frac{\gamma_t(x+\epsilon)-\gamma_t(x-\epsilon)}{2\epsilon}\\ &\qquad< \lim_{\epsilon\to 0}\frac{r\varrho_t(x+r)S_3+r\varrho_t(x-r)S_1+2r\max\{\varrho_t(x+r),\varrho_t(x-r)\}S_2}{(S_1+S_2)(S_3+S_2)}\\ &\qquad\qquad< \lim_{\epsilon\to 0}\frac{S_2(S_1+S_3)+S_2^2}{(S_1+S_2)(S_3+S_2)} < 1.\end{aligned}$$

Consequently, according to (4.5), $\rho_t(x) < \rho_{t+1}\big(\gamma_t(x)\big)$. ■

Lemma 4.3 is employed in the study of how population clusters are formed and diverge away from the center of a weighted Dirac delta input, presented in the next section.

5. Exogenous input. In this section, we study information assimilation under a single constant input centered at the advertised opinion with a uniformly distributed initial population. As discussed in the introduction, one of the most influential inputs in the evolution of opinions is the mainstream media. With the increase in the popularity of new communication technologies such as blogging and tweeting, the message sent by the media is restated publicly with bias, and thus the direct influence of media on the public has been replaced by a two-way relationship [30]. Accordingly, an exogenous input in an Eulerian HK system can be modeled as a background normal distribution centered at the opinion of an expert with the total area under the curve equal to $w \in \mathbb{R}_{>0}$. The variance of this input depends on various factors such as biased repetition of the message by blogs, and w depends on factors such as how strongly the message is being reported. For such an input in an Eulerian HK system, we introduce the input's "attracted population," which is the total population that is attracted to the input's center as time goes to infinity. Here we study two families of exogenous inputs. First, we assume that there is no bias in the rebroadcast by blogs and tweets. Hence, the input can be represented by a Dirac delta distribution, and we compute an upper bound for this input's attracted population. Second, we consider a normally distributed input which influences only a finite range of opinions (i.e., a truncated normal distribution) and present our conjecture based on simulation results.

Definition 5.1 (attracted population). *Consider an Eulerian HK system with mass distribution μ_t and a time-invariant weighted Dirac delta input $u = w\delta_{\hat{x}}$ for $\hat{x} \in \operatorname{supp}\mu_t$ and weight $w \in \mathbb{R}_{>0}$. The input's* attracted population, *denoted by $\mathcal{A}(r,u,\mu_0) \in [0, \mu_t(\operatorname{supp}\mu_t)]$, is defined by*

$$\mathcal{A}(r,u,\mu_0) = \lim_{\epsilon\to 0}\lim_{t\to\infty}\mu_t\big((\hat{x}-\epsilon,\hat{x}+\epsilon)\big).$$

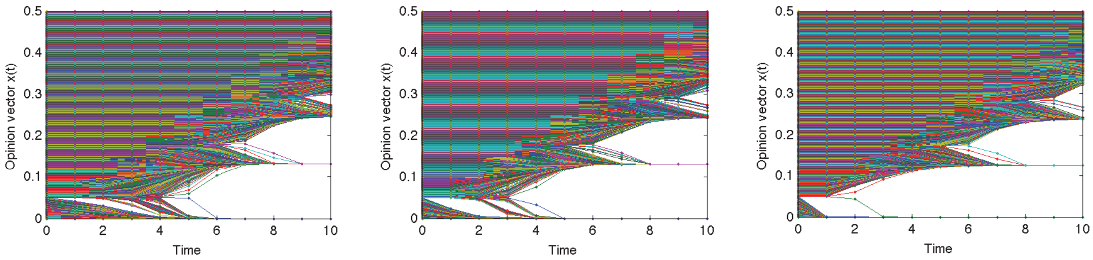

Figure 3. *Evolutions of Eulerian HK systems with uniform initial distribution* $\mu_0 \sim \mathcal{U}(-L, L)$ *and input* $u = w\delta_0$ *are illustrated. We consider* $\rho_0 = 1$, $L \in \mathbb{R}_{>0}$ *and sufficiently larger than* $r = 0.05$, *and three values of* $\frac{r}{r+w} = 0.7, 0.3, 0.01$ *(from left to right). To ensure lucidity of results, the agent based behavior of discretized Eulerian HK systems is represented. Simulating this system for various values of* $\frac{r}{r+w}$, *we conjecture that Assumption 5.1 holds for all* $t \geq 5$ *and Assumption 5.2 is always satisfied. Moreover, it can be observed that for a larger* w *(i.e., a more strict report of the message) a smaller population will be attracted to the input's center and the rest will cluster away from the input.*

Now, we consider the effect of the weighted Dirac delta input $w\delta_0$ on an initially uniform distribution of population over the opinion space $\mathbb{R}$. Note that the influence of a Dirac delta input on an infinite opinion space is roughly equivalent to its influence on the center of a bounded opinion interval whose length is sufficiently larger than r such that the impact of the opinion space's boundaries can be ignored. Furthermore, for a constant ratio ρ_0/w, where ρ_0 denotes the initial mass density, one may verify that the evolution of the mass distribution is independent of ρ_0. For such an Eulerian HK system with input, if $\partial\Delta_t(x)/\partial x$ exists for all $x > r$ at time t, then we define

$$\xi(t) = \inf\left\{x : \frac{\partial\Delta_t(x)}{\partial x} = 0 \text{ and } \gamma_t(x) > r\right\}.$$

Moreover, we denote $\frac{r}{r+w}$ by α and the opinions $\gamma_t \circ \cdots \circ \gamma_0(r)/r$ and $\gamma_t \circ \cdots \circ \gamma_1(\sqrt{1+\alpha}\, r)$ by $c(t)$ and $\eta(t)$, respectively. Later we show that $c(t)r$ is the largest opinion of the central population whose initial opinion belongs to $[-r, r]$ and $\eta(1)$ is a fixed point of $\gamma_1(x)$. For simplicity of the analysis, we employ the following technical assumptions (that are supported by numerical results) in Theorem 5.1.

Assumption 5.1. *If* $\xi(t)$ *is finite, then* $\gamma_t(\xi(t)) \geq 2r$, $\Delta_t(\xi(t)) \geq 0$, *and* $\gamma_t(\xi(t)) \geq \xi(t+1)$.

Assumption 5.2. *For* $t \in \{0, \ldots, 5\}$, $y_t((0, 2r)) < r$, *and for* $t = 5$, $\gamma_t(r/2) < r/2$, $\eta(t) > \gamma_t(\xi(t))$, $\mu_t((c(t), 2r)) < (\sqrt{1+\alpha} - 1)r$, *and if* $\alpha < 0.75$, *then* $c(5) < 0.5$.

We explain in Remark 1 that Assumption 5.1 is based upon the conjecture that the inter-cluster distance in the Eulerian HK model of opinion dynamics with uniform initial condition is greater than or equal to $2r$, and $\xi(t)$ separates the attracted population from a cluster that is formed above this population. Assumption 5.2 is based on the behavior of the system in the first five time steps and is numerically conjectured; see the simulation results for different values of α in Figure 3.

Theorem 5.1 (upper bound on attracted population). *Consider an Eulerian HK system with uniform initial mass distribution* μ_0 *with* $\rho_0(x) = \rho_0 = 1$ *for any* $x \in \mathbb{R}$ *and a time-invariant input* $u = w\delta_0$, *where* $w \in \mathbb{R}_{>0}$. *If the system satisfies Assumption 5.2,* $\frac{r}{r+w} < 0.75$, *and Assumption 5.1 holds for all* $t \geq 5$, *then*

(i) $\mu_t \ll \mathcal{L}^1$ *for all* $t \geq 0$, *and*

(ii) $\mathcal{A}(r, u, \mu_0) < 2r\sqrt{1 + \frac{r}{r+w}}$.

Proof. Due to the symmetry of the population with respect to the origin, throughout this proof, we focus only on the dynamics of population with positive opinions. Note that the effect of the input on the mass distribution propagates with rate r. In other words, at any time t, the mass distribution over opinion interval (tr, ∞) remains uniform with density ρ_0.

Consider the following statements:

(a) $\mu_t \ll \mathcal{L}^1$,

(b) $\rho_t(x) > 0$ and is continuous for all $x \in (r, \infty)$,

(c) $y_t((0, 2r)) < r$,

(d) $\gamma_t(r/2) < r/2$, and

(e) $\mu_t((c(t), 2r)) < (\sqrt{1+\alpha} - 1)r$.

We depict the interconnection between the statements above, on which we establish the theorem. More specifically, statements (a) and (e) construct parts (i) and (ii) of the theorem, respectively. First, it is easy to see that statements (a) and (b) hold at $t = 0$. Based on Assumption 5.2, statement (c) holds for $t \in \{0, \ldots, 5\}$. Here we prove by induction that at any time $t \geq 0$, if the three statements hold true, then statements (a) and (b) hold true at $t+1$. Second, we show that if all these statements hold at any time $t \geq 5$, then statements (c), (d), and (e) will hold true at $t+1$.

Regarding statement (a), denote the mass distribution μ_t over interval $(r, (t+1)r]$ at time $t \geq 0$ by $\hat{\mu}_t$. Then, the mass distributions over intervals $[0, r]$ and $((t+1)r, (t+2)r]$ at time t can be regarded as inputs for $\hat{\mu}_t$ which are absolutely continuous. Hence, the system with initial mass distribution $\hat{\mu}_t$ with support $(r, (t+1)r]$ satisfies the sufficient conditions of Theorem 3.1. Therefore, $\hat{\mu}_{t+1}$ with support $\lim_{\epsilon \to 0}[\gamma_t(r+\epsilon), (t+1)r]$ is absolutely continuous, where $\gamma_{t+1}((t+1)r) = (t+1)r$. Since statements (b) and (c) hold at time t, $\lim_{\epsilon \to 0} \gamma_t(r + \epsilon) < r$, and thus it can be shown that $\gamma_t(x)$ satisfies the strict monotonic behavior stated in Theorem 3.1 part (ii) for all $x \in \mathbb{R}$. In view of the flow map's strict monotonicity, the mass distribution over opinion interval $[\gamma_t(-r), \gamma_t(r)]$ at time $t+1$ contains no atom and thus is absolutely continuous; however, its support does not necessarily equal $[\gamma_t(-r), \gamma_t(r)]$ and may consist of a set of separate intervals.

Regarding statement (b), as shown above, $\operatorname{supp} \mu_{t+1}$ over interval $[r, \infty)$ is a closed interval, and since $\lim_{\epsilon \to 0} \gamma_t(r+\epsilon) < r$ for $t \geq 0$, this support is equal to $[r, \infty)$. Therefore, $\rho_{t+1}(x) > 0$ for all $x \in (r, \infty)$, and based on Lemma 4.3 part (i), ρ_{t+1} is continuous for all $x \in (r, \infty)$.

Regarding statement (c), since statement (e) holds at time $t \geq 5$, we have

$$y_{t+1}((0, 2r)) < \frac{c(t)r^2 + 2r^2(\sqrt{1+\alpha} - 1)}{r\sqrt{1+\alpha}}. \tag{5.1}$$

Assumption $c(5) < 0.5$ together with statement (d) at time t results in $c(t) < 0.5$; therefore, $y_{t+1}((0, 2r)) < r$.

Regarding statement (d), similarly, we have

$$\gamma_t(r/2) < \frac{(\sqrt{1+\alpha} - 1)r(c(t) + r)}{w + 2r + (\sqrt{1+\alpha} - 1)r}. \tag{5.2}$$

Since $w = r/\alpha - r$ and $c(t) < 0.5$, the above inequality results in

$$\gamma_t(r/2) < \frac{(\sqrt{1+\alpha}-1)1.5r}{1/\alpha + \sqrt{1+\alpha}} < r/2. \tag{5.3}$$

Regarding statement (e), first, as shown above, statement (b) holds at times $t \geq 0$ and $t+1$; hence Lemma 4.3 part (ii) leads to the existence and continuity of $\partial\Delta_t(x)/\partial x$ for all $x \in (r, \infty)$. Moreover, according to Theorem 3.1 part (ii), $\gamma_t(r) < r$ for $t \geq 5$ results in $\gamma_t^{-1}(r) > r$, and thus $\Delta_t(\gamma_t^{-1}(r)) < 0$. Since the mass distribution for $x > tr$ is not affected by input at any time t, $\Delta_t(x) = 0$ for $x \in (tr, \infty)$, which shows that $\xi(t)$ is finite. Now, from $\Delta_t(\xi(t)) > 0$ and $\Delta_t(\gamma_t^{-1}(r)) < 0$, since $\partial\Delta_t(x)/\partial x$ is continuous over (r, ∞) and $\xi(t)$ is defined as the infimum of all $x > r$ such that $\partial\Delta_t(x)/\partial x = 0$ and $\gamma_t(x) > r$, we have $\partial\Delta_t(x)/\partial x > 0$ for all $x \in (\gamma_t^{-1}(r), \xi(t))$. Therefore, according to Lemma 4.3 part (ii), $\rho_{t+1}\big(\gamma_t(x)\big) < \rho_t(x)$ for all $x \in (\gamma_t^{-1}(r), \xi(t))$. Equivalently, $\rho_{t+1}(x) < \rho_t\big(\gamma_t^{-1}(x)\big)$ for all $x \in (r, \gamma_t(\xi(t)))$. Since according to Assumption 5.1, $(r, 2r) \subseteq (r, \gamma_t(\xi(t)))$, we have $\mu_{t+1}((r, 2r)) < \mu_t((r, 2r))$.

Finally, part (i) of the theorem can be concluded from statement (a), which holds for all $t \geq 0$. Regarding part (ii), from Assumption 5.2, statements (d) and (e) hold at $t = 5$, and, as shown in the beginning of this proof, statements (a), (b), and (c) hold at $t \in \{1, \ldots, 5\}$. Therefore, by induction, all statements will hold for all $t \geq 5$. Now, since statements (a) and (b) hold true, part (ii) of Theorem 3.1 together with $\eta(5) > \gamma_5(\xi(5))$ implies that $\eta(t) > 2r$ for all $t \geq 5$, which results in

$$\begin{aligned}\mathcal{A}(r, u, \mu_0) < \mu_t((-2r, 2r)) < \mu_t\big((-\eta(t), \eta(t))\big) &= \mu_1\big((-\eta(1), \eta(1))\big) \\ &= \rho_0\left(2r + 2r\left(\sqrt{1+\frac{r}{r+w}} - 1\right)\right) = 2r\sqrt{1+\frac{r}{r+w}}. \qquad \blacksquare\end{aligned}$$

Remark 1. *Assumption 5.1 stems from the conjecture that in the Eulerian HK system of Theorem 5.1, a cluster forms above opinion $2r$ for $t \geq 5$. A cluster, roughly speaking, is any opinion interval (a, b) with a mass density higher than the density over intervals $(a-r, a)$ and $(b, b+r)$. This property, according to Lemma 4.3 part (iii), leads to an increase in the cluster's density in the next iteration. Hence, according to Lemma 4.3 part (ii), $\partial\Delta_t(x)/\partial x < 0$ over the cluster, and $\partial\Delta_t(x)/\partial x > 0$ above and below the cluster, which leads to the existence of opinions with $\partial\Delta_t(x)/\partial x = 0$ above and below the cluster.*

Theorem 5.1 illustrates that a Dirac delta input, which represents a nonbiased rebroadcast of an expert's opinions, can attract a proportion of the population whose opinions lie in more than r and less than $2r$ distance from the input's center. Furthermore, the size of the attracted population has an inverse relation with w, which is the total area under the input's distribution (i.e., the input's intensity). The interpretation of this result is that a strong message can attract a smaller population than a mild message broadcast by media.

Next, we consider the case of a truncated normal distribution as the input for the Eulerian HK system, which represents a biased repetition of the media's message by blogs that influence only a finite range of opinions. Note that a nontruncated normal distribution with a nonzero variance as an input eventually derives the whole population into a consensus. We simulated Eulerian HK systems with uniform initial distribution $\mu_0 \sim \mathcal{U}(-x_0, x_0)$ and input $u \sim \mathcal{N}(\hat{x}, \sigma^2)$ with support $[-3\sigma, 3\sigma]$ for different values of the following parameters: σ, x_0,

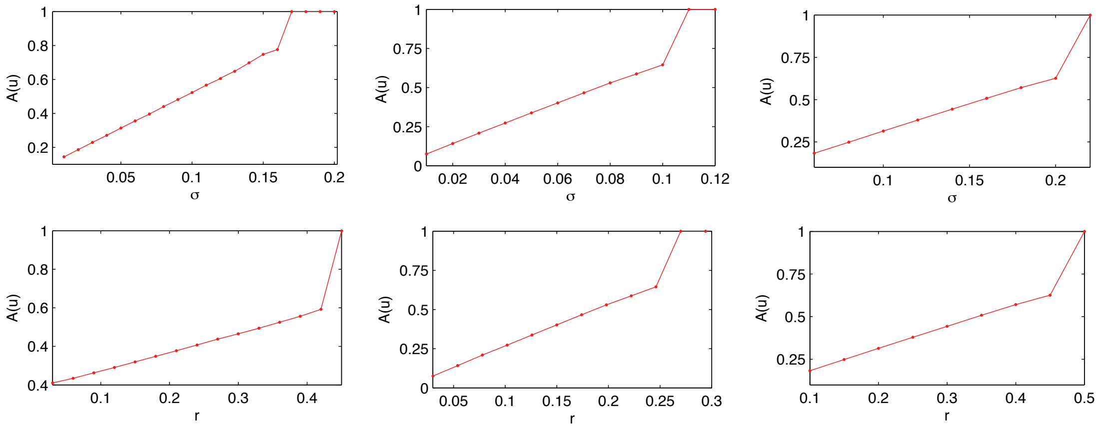

Figure 4. *In evolutions of Eulerian HK systems with uniform initial distribution $\mu_0 \sim \mathcal{U}(-x_0, x_0)$ and input $u \sim \mathcal{N}(0, \sigma^2)$, the attracted population $\mathcal{A}(r, u, \mu_0)$ is found for different values of the following parameters: σ, x_0, and r. In conclusion, for $\sigma \le 0.05|\operatorname{supp}\mu_0|$ and $r \le 0.125|\operatorname{supp}\mu_0|$, the simulations reveal the following linear relation: $\mathcal{A}(r, u, \mu_0) \simeq (8\sigma + 2r)/|\operatorname{supp}\mu_0|$. In the top left plot, $x_0 = 1$, $r = 0.1$, and $\sigma \in \{0.01, 0.02, \dots, 0.17\}$; in the bottom left plot, $x_0 = 1$, $\sigma = 0.04$, and $r \in \{0.03, 0.06, \dots, 0.45\}$; in the middle plot, $x_0 = 1$ and $(\sigma, r) \in \{(0.01, 0.03), \dots, (0.12, 0.3)\}$; and in the right plot, $x_0 = 2$ and $(\sigma, r) \in \{(0.06, 0.1), \dots, (0.22, 0.5)\}$.*

and confidence bound r; see Figure 4. We observe that the attracted population is larger than that for the case of nonbiased advertising. More specifically, for $\sigma \le 0.05|\operatorname{supp}\mu_0|$, $r \le 0.125|\operatorname{supp}\mu_0|$, and $\hat{x}$ such that $\{\hat{x} - \frac{4\sigma+r}{|\operatorname{supp}\mu_0|}, \hat{x} + \frac{4\sigma+r}{|\operatorname{supp}\mu_0|}\} \in \operatorname{supp}\mu_0$, the simulations reveal the following linear relation:

$$\mathcal{A}(r, u, \mu_0) \simeq \frac{8\sigma + 2r}{|\operatorname{supp}\mu_0|}.$$

Therefore, we conjecture that the attracted population is approximately an increasing affine function of the population's confidence bound and of the input's standard deviation; see Figure 4. This result suggests that (i) a higher biased rebroadcast of media's message by various blogs and/or (ii) a larger confidence bound result in the attraction of a larger population to the advertised message. Moreover, the blogs' bias has a higher effect than the population's confidence bound in the attraction of the population to the advertised opinion.

6. Conclusion. In this paper, we studied the formation of opinions in a large population that is governed by endogenous (interactions with peers) and exogenous (influence of media) factors. We focused on an Eulerian bounded-confidence model of opinion dynamics and proposed a reasonable model for exogenous inputs. We proved mathematical properties of this system's dynamics with time-varying input and derived a simple sufficient condition for opinion consensus. In particular, for the case of no exogenous input, we showed the convergence of the population's distribution to a sum of weighted Dirac delta distributions. To analyze the information assimilation in our system, we modeled the exogenous inputs as background weighted Dirac delta and (truncated) normal distributions centered at the opinion of an expert and defined the *attracted population* of a constant input. For the case of weighted Dirac delta input, and uniformly distributed initial population, under technical assumptions, we established an upper bound on the attracted population. This upper bound is an increasing function of population's confidence bound and a decreasing function of the input's measure. For the case of a normally distributed input with truncated support, we conjectured that the attracted population is approximately an increasing affine function of the population's confidence bound and of the input's standard deviation.

There are three main future challenges. First, we plan to complete the study on an Eulerian HK system with Dirac delta input; we aim to fully understand the time-invariant input's influence on the overall population and on the eventual emergence of clusters in equilibria. Second, we intend to focus on the mathematical analysis of the more general case, where a normally distributed input is considered. Third, we are interested in strategic opinion manipulation, and it is of great importance to study how a time-varying input has a higher efficiency in manipulation of opinions than a constant input.

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